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Further Improvements in Waring's Problem. III Eighth Powers

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Further improvements in Waring's problem. III Eighth powers

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In this paper we continue our development of the methods of Vaughan & Wooley, these being based on the use of exponential sums over integers having only small prime divisors. On this occasion we concentrate on improvements in the estimation of the contribution of the major arcs arising in the efficient differencing process. By considering the underlying diophantine equation, we are able to replace certain smooth Weyl sums by classical Weyl sums, and thus we are able to utilize a number of pruning processes to facilitate our analysis. These methods lead to improvements in Waring's problem for larger k. In this instance we prove that $G(8) \leqslant 42$, which is to say that all sufficiently large natural numbers are the sum of at most 42 eighth powers of integers. This improves on the earlier bound $G(8) \leqslant 43$.

1. Introduction

As usual we define $G(k)$ to be the least number s such that every sufficiently large natural number is the sum of, at most, s kth powers of natural numbers. In this paper we continue our development of the methods of Vaughan & Wooley (1993), which hereinafter we abbreviate to FIWP. Generally, following Vaughan (1989), our methods are dependent on upper bounds for the number, $S_s^{(k)}(P,R)$, of solutions of the diophantine equations

$$x_1^k+\ldots+x_s^k = y_1^k+\ldots+y_s^k, \tag{1.1}$$

with $x_i, y_i \in \mathscr{A}(P,R)$, where throughout we write

$$\mathscr{A}(P,R) = \{1 \leqslant n \leqslant P : p \text{ prime}, p \mid n \text{ implies } p \leqslant R\}.$$

In FIWP we are preoccupied with refinements of the efficient differencing process initiated in Wooley (1992). When $k=8$ the methods fail to give bounds for $S_s^{(k)}(P,P^\eta)$ suitable for the establishment of the theorem below, by a power of P. The source of this failure is the inherent difficulty of estimating the contribution from the major arcs arising in the efficient differencing process.

We are able to make further progress by applying a pruning process, based on the exploitation of the well-understood behaviour of classical Weyl sums on suitable 'major arcs'. To ascend to a position from which such exploitation is possible, we make the crucial observation that at a certain point in the efficient differencing process, two smooth Weyl sums may be replaced by classical Weyl sums with no significant loss. Thus, in §3, we develop effective treatments for the various mean values which arise in the pruning processes. For larger k, these methods prove to be of importance in applying the methods of FIWP to the task of obtaining an upper bound for $G(k)$, although in general, further ideas are required to obtain the strongest results deriving from this circle of ideas. We therefore confine ourselves here to the case $k = 8$. In FIWP it is shown that $G(8) \leqslant 43$. We are now able to establish the following theorem.

Theorem. $G(8) \leqslant 42$.

Since the methods of this paper are closely related to those of FIWP (see also Vaughan, this volume), we shall find it convenient to use the notation introduced there.

2. Preliminary observations

We first recall some of the notation of FIWP. Throughout, k will denote an arbitrary integer exceeding 2, s will denote a positive integer, and ϵ and η will denote sufficiently small positive numbers. We take P to be a large positive real number depending at most on k, s, ϵ and η. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on k, s, ϵ and η. We make frequent use of vector notation for brevity. For example, $(c_1, \ldots, c_t)$ is abbreviated to $\boldsymbol{c}$. Also, we shall write $e(\alpha)$ for $\mathrm{e}^{2\pi i \alpha}$, and $[x]$ for the greatest integer not exceeding x.

In an effort to simplify our analysis, we adopt the following convention concerning the numbers ϵ and R. Whenever ϵ or R appear in a statement, either implicitly or explicitly, we assert that for each $\epsilon > 0$, there exists a positive number $\eta_0(\epsilon, s, k)$ such that the statement holds whenever $R = P^\eta$, with $0 < \eta \leqslant \eta_0(\epsilon, s, k)$. Note that the 'value' of ϵ, and η_0, may change from statement to statement, and hence also the dependency of implicit constants on ϵ and η. Since our iterative methods will involve only a finite number of statements (depending at most on k, s and ϵ), there is no danger of losing control of implicit constants through the successive changes implicit in our arguments. Finally, we use the symbol $\approx$ to indicate that constants and powers of R and P^ϵ are to be ignored.

For each $s \in \mathbb{N}$ we take $\phi_i = \phi_{i,s}$ $(i = 1, \ldots, k)$ to be real numbers, with $0 \leqslant \phi_i \leqslant 1/k$, to be chosen later. We then take

$$P_j = 2^j P, \quad M_j = P^{\phi_j}, \quad H_j = P_j M_j^{-k}, \quad Q_j = P_j (M_1 \ldots M_j)^{-1} \quad (1 \leqslant j \leqslant k).$$

We also write
$$\tilde{H}_j = \prod_{i=1}^{j} H_i \quad \text{and} \quad \tilde{M}_j = \prod_{i=1}^{j} M_i R.$$

We define the modified forward difference operator, Δ_1^*, by

$$\Delta_1^*(f(x); h; m) = m^{-k}(f(x + hm^k) - f(x)),$$

and define Δ_j^* recursively by

$$\Delta_{j+1}^*(f(x); h, \ldots, h_{j+1}; m_1, \ldots, m_{j+1}) = \Delta_1^*(\Delta_j^*(f(x); h_1, \ldots, h_j; m_1, \ldots, m_j); h_{j+1}; m_{j+1}).$$

We also adopt the convention that $\Delta_0^*(f(x); h; m) = f(x)$.

For $0 \leqslant j \leqslant k$ let

$$\Psi_j = \Psi_j(z; h_1, \ldots, h_j; m_1, \ldots, m_j) = \Delta_j^*(f(z); 2h_1, \ldots, 2h_j; m_1, \ldots, m_j),$$

where $f(z) = (z - h_1 m_1^k - \ldots - h_j m_j^k)^k$.

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Write $$f_j(\alpha)=\sum_{x\in\mathscr{A}(Q_j,R)} e(\alpha x^k) \quad\text{and}\quad g_j(\alpha)=\sum_{1\leqslant x\leqslant Q_j} e(\alpha x^k).$$

Also, write $$F_j(\alpha)=\sum_{z,\boldsymbol{h},\boldsymbol{m}} e(\alpha\Psi_j(z;\boldsymbol{h};\boldsymbol{m})),$$

where the summation is over $z,\boldsymbol{h},\boldsymbol{m}$ with

$$1\leqslant z\leqslant P_j,\quad M_i<m_i\leqslant M_iR,\quad m_i\in\mathscr{A}(P,R),\quad 1\leqslant h_i\leqslant 2^{j-i}H_i,\quad (1\leqslant i\leqslant j).$$

We define $S_s^{(k)}(P,R)$ as in the introduction, and, when no confusion is possible, we shall suppress the superscript k. Suppose that the real numbers λ_s $(1\leqslant s<\infty)$ have the property that

$$S_s^{(k)}(P,R)\ll P^{\lambda_s+\epsilon}. \tag{2.1}$$

Such numbers certainly exist, since we may trivially take $\lambda_s=2s$. Then, for each s, we define the quantity Δ_s by

$$\lambda_s=2s-k+\Delta_s. \tag{2.2}$$

At the core of our argument is the use of a modified version of Lemma 2.2 of FIWP, which we now record.

Lemma 2.1. *Whenever $0<t<s$ and $1\leqslant j\leqslant k-1$, we have*

$$\int_0^1 |F_j(\alpha)f_j(\alpha)^{2s}|\,d\alpha\ll P^{\epsilon}(Q_j^{\lambda_t})^{\frac{1}{2}}(\tilde{H}_j\tilde{M}_jM_{j+1}^{4s-2t-1}T_{j+1})^{\frac{1}{2}},$$

where $$T_{j+1}=P\tilde{H}_j\tilde{M}_{j+1}Q_{j+1}^{\lambda_{2s-t}}+\int_0^1|F_{j+1}(\alpha)g_{j+1}(\alpha)^2f_{j+1}(\alpha)^{4s-2t-2}|\,d\alpha. \tag{2.3}$$

Proof. The proof is almost identical to that of Lemma 2.2 of FIWP. We observe that in eqn (3.5) of Wooley (1992), an upper bound for the quantity U_1 appearing in the proof of Lemma 3.1 of that paper is obtained by relaxing the restriction on x_1 and y_1, so that only $1\leqslant x_1,\ y_1\leqslant Q_{j+1}$. The lemma then follows as before, on considering the underlying diophantine equations.

Our argument will be based on a Hardy–Littlewood dissection, together with a suitable pruning operation. We now describe the various sets of arcs which we shall make use of. Here and throughout, we write $w=2^{1+j-k}$.

Definition 2.2. *Suppose that $1\leqslant j\leqslant k-4$.*

(i) *Let $\mathfrak{m}_j$ denote the set of points in $[0,1]$ with the property that whenever there are $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ with $(a,q)=1$, and*

$$qP^{-1}Q_j^kR^{k(k-j)}|\alpha-a/q|\leqslant 1, \tag{2.4}$$

then $q>P$. Further, let $\mathfrak{M}_j=[0,1]\backslash\mathfrak{m}_j$.

(ii) *When $(q,a)=1$, let $\mathfrak{M}_j(q,a)$ be the set of α in $[0,1]$ for which (2.4) holds.*

(iii) *Let $\mathfrak{n}_j$ denote the set of points in $\mathfrak{M}_j$ with the property that whenever there are $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ with $(a,q)=1$, and*

$$q(PM_1)^{-w(k-j)}Q_j^k|\alpha-a/q|\leqslant 1, \tag{2.5}$$

then $q>(PM_1)^{w(k-j)}$. Further, let $\mathfrak{N}_j=\mathfrak{M}_j\backslash\mathfrak{n}_j$.

(iv) *When $(q,a)=1$, let $\mathfrak{N}_j(q,a)$ be the set of α in $\mathfrak{M}_j$ for which (2.5) holds.*

We note that the $\mathfrak{M}_j(q,a)$ with $0\leqslant a\leqslant q\leqslant P$ are disjoint, and also that the $\mathfrak{N}_j(q,a)$ with $0\leqslant a\leqslant q\leqslant(PM_1)^{w(k-j)}$ are disjoint.

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Finally, we shall record the definitions of some generating functions of use on the various major arcs. We write

$$S(q,a)=\sum_{r=1}^{q} e(ar^k/q), \quad \text{and} \quad v_j(\beta)=\sum_{1\leqslant x\leqslant Q_j^k}\frac{1}{k}x^{1/k-1}e(\beta x),$$

and $$V_j(\alpha;q,a)=q^{-1}S(q,a)\,v_j(\alpha-a/q).$$

We then define $g_j^*(\alpha)$ to be zero whenever $\alpha\in\mathfrak{n}_j$, and by $g_j^*(\alpha)=V_j(\alpha;q,a)$ whenever $\alpha\in\mathfrak{N}_j(q,a)$ and $0\leqslant a\leqslant q\leqslant(PM_1)^{w(k-j)}$. Also, we write

$$\tau_j(q,a,\boldsymbol{h},\boldsymbol{m})=\left|\sum_{r=1}^{q} e\left(\frac{a}{q}\Psi_j(r,\boldsymbol{h},\boldsymbol{m})\right)\right|.$$

We then define $F_j^*(\alpha)$ to be zero whenever $\alpha\in\mathfrak{m}_j$, and by

$$F_j^*(\alpha)=\sum_{\boldsymbol{m}}\sum_{\boldsymbol{h}}\frac{Pq^{-1}\tau_j(q,a,\boldsymbol{h},\boldsymbol{m})}{(1+|\alpha-a/q|\,h_1\ldots h_j P^{k-j})^{1/(k-j)}},$$

whenever $\alpha\in\mathfrak{M}_j(q,a)$ and $0\leqslant a\leqslant q\leqslant P$.

3. A refined Hardy–Littlewood dissection for larger k

We now describe the various pruning operations which underpin our argument, beginning with a lemma which reduces the problem of estimating the integral in (2.3) to one of estimating the mean value $I_{j+1,\,2s-t-1}$ which we define by

$$I_{j,u}=\int_{\mathfrak{N}_j} F_j^*(\alpha)\,|g_j^*(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha.$$

Lemma 3.1. *Suppose that* $1\leqslant j\leqslant k-4$. *Let* u *be a positive integer, and define*

$$t=\left[\left(\frac{k-j+1}{k-j}\right)u+1\right],\quad \theta=t-\left(\frac{k-j+1}{k-j}\right)u,$$

and $$\nu_u=\frac{k-j}{k-j+1}(\theta\Delta_{t-1}+(1-\theta)\Delta_t).$$

Then $$\int_0^1 |F_j(\alpha)\,g_j(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha \ll P^{1+\epsilon}\tilde{H}_j\tilde{M}_j Q_j^{2u+2-k}\mathscr{M}+I_{j,u},$$

where $$\mathscr{M}=(PM_1)^{-w}Q_j^{\Delta_{u+1}}+(PM_1)^{(k-j)w}Q_j^{\nu_u-2}.$$

Proof. In view of Definition 2.2, we may imitate the analysis of the proof of Lemma 13.1 of FIWP to deduce that

$$\int_0^1 |F_j(\alpha)\,g_j(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha \ll I_1+I_2, \tag{3.1}$$

where $$I_1=\int_{\mathfrak{M}_j} F_j^*(\alpha)\,|g_j(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha, \tag{3.2}$$

and $$I_2=\left(P^{(k-j-1)/(k-j)+\epsilon}\tilde{H}_j\tilde{M}_j+\sup_{\alpha\in\mathfrak{m}_j}|F_j(\alpha)|\right)\int_0^1|g_j(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha. \tag{3.3}$$

We first consider I_2. By using a Weyl differencing argument, we may follow the pattern established in Lemmata 6.1 and 12.1 of FIWP to deduce, from Lemma 4.1 and Corollary 4.2.1 of that paper, that

$$\sup_{\alpha\in\mathfrak{m}_j}|F_j(\alpha)| \ll P^{1+\epsilon}\tilde{H}_j\tilde{M}_j(PM_1)^{-w}. \tag{3.4}$$

Further, when $k-j \geqslant 4$ we have

$$2^{k-j-1}/(k-j) \geqslant 2 > 1+\phi_1,$$

and hence
$$P^{(k-j-1)/(k-j)}\tilde{H}_j\tilde{M}_j \ll P\tilde{H}_j\tilde{M}_j(PM_1)^{-w}. \tag{3.5}$$

Also, on noting the observation at the end of §3 of Wooley (1992),

$$\int_0^1 |g_j(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha \ll Q_j^{\lambda_{u+1}+\epsilon}. \tag{3.6}$$

Thus, in view of (2.2), we obtain from (3.4)–(3.6) the estimate

$$I_2 \ll P^{1+\epsilon}\tilde{H}_j\tilde{M}_j\, Q_j^{2u+2-k}(PM_1)^{-w}\, Q_j^{\Delta_{u+1}}. \tag{3.7}$$

We now prune the arcs $\mathfrak{M}_j$ occurring in I_1 down to the arcs $\mathfrak{N}_j$. When $\alpha\in\mathfrak{n}_j\cap\mathfrak{M}_j(q,a)$, by Definition 2.2(iii) we have

$$q+Q_j^k|q\alpha-a| > (PM_1)^{w(k-j)}.$$

Hence by Lemmata 4.7 and 4.8 of FIWP,

$$\begin{aligned}\sup_{\alpha\in\mathfrak{n}_j\cap\mathfrak{M}_j(q,a)} F_j^*(\alpha) &\ll \sum_{\mathbf{m}}\sum_{\mathbf{h}} P(q+P^{k-j}h_1\dots h_j\,|q\alpha-a|)^{-1/(k-j)}\\ &\ll P\tilde{H}_j\tilde{M}_j(q+Q_j^k\,|q\alpha-a|)^{-1/(k-j)}\\ &\ll P\tilde{H}_j\tilde{M}_j(PM_1)^{-w}.\end{aligned}$$

Thus as above we deduce from (3.2) that

$$I_1 \ll I_3 + P^{1+\epsilon}\tilde{H}_j\tilde{M}_j\, Q_j^{2u+2-k}(PM_1)^{-w}\, Q_j^{\Delta_{u+1}}, \tag{3.8}$$

where
$$I_3 = \int_{\mathfrak{N}_j} F_j^*(\alpha)\,|g_j(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha.$$

By Vaughan (1981*a*, Theorem 2), when $\alpha\in\mathfrak{N}_j(q,a)$ we have

$$g_j(\alpha)-g_j^*(\alpha) \ll (q+Q_j^k\,|q\alpha-a|)^{\frac{1}{2}+\epsilon} \ll (PM_1)^{\frac{1}{2}w(k-j)+\epsilon}.$$

Thus
$$I_3 \ll (PM_1)^{w(k-j)+\epsilon} J_1 + \int_{\mathfrak{N}_j} F_j^*(\alpha)\,|g_j^*(\alpha)^2 f_j(\alpha)^{2u}|\,\mathrm{d}\alpha, \tag{3.9}$$

where
$$J_1 = \int_{\mathfrak{N}_j} F_j^*(\alpha)\,|f_j(\alpha)|^{2u}\,\mathrm{d}\alpha.$$

But by Hölder's inequality,

$$J_1^{k-j+1} \ll L_1 K_{t-1}^{(k-j)\theta} K_t^{(k-j)(1-\theta)},$$

where
$$K_s = \int_0^1 |f_j(\alpha)|^{2s}\,\mathrm{d}\alpha \ll Q_j^{\lambda_s+\epsilon} \quad (s=t-1,t), \tag{3.10}$$

and
$$L_1 = \int_0^1 F_j^*(\alpha)^{k-j+1}\,\mathrm{d}\alpha. \tag{3.11}$$

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But on applying Lemma 4.10 of FIWP, we obtain

$$L_1 \ll P^{\epsilon}(P\tilde{H}_j \tilde{M}_j)^{k-j+1} Q_j^{-k}, \tag{3.12}$$

and hence we deduce that

$$J_1 \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j Q_j^{2u-k+\nu_u}. \tag{3.13}$$

The lemma now follows on combining (3.7)–(3.9) and (3.13).

There are a number of approaches to the task of estimating the integral $I_{j,u}$ appearing in the statement of Lemma 3.1, the effectiveness of the respective methods depending on the relative sizes of Q_j and $(PM_1)^{w(k-j)}$. Although we shall require only a relatively simple approach in our applications, we explain two ideas, so as to increase the flexibility of our methods. The first idea is simply to apply Hölder's inequality.

Lemma 3.2. *Suppose that $1 \leqslant j \leqslant k-4$. Let u be a positive integer, and define*

$$\gamma = 1 - \frac{1}{k-j+1} - \frac{2}{k+1}, \quad t = [\gamma^{-1}u+1], \quad \theta = t-\gamma^{-1}u,$$

and
$$\rho_u = \gamma(\theta\Delta_{t-1} + (1-\theta)\,\Delta_t).$$

Then
$$I_{j,u} \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j Q_j^{2u+2-k+\rho_u},$$

Proof. By Hölder's inequality,

$$I_{j,u} \leqslant L_1^{1/(k-j+1)} L_2^{2/(k+1)} K_{t-1}^{\gamma\theta} K_t^{\gamma(1-\theta)}, \tag{3.14}$$

where K_s $(s = t-1, t)$ and L_1 are given by (3.10) and (3.11), and

$$L_2 = \int_{\mathfrak{N}_j} |g_j^*(\alpha)|^{k+1}\,\mathrm{d}\alpha. \tag{3.15}$$

The methods of Vaughan (1981*b*, section 4.4) yield $L_2 \ll Q_j^{1+\epsilon}$. Thus by (3.12) and (3.14), we obtain

$$I_{j,u} \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j Q_j^{2-k} (Q_j^{\lambda_{t-1}+k})^{\gamma\theta} (Q_j^{\lambda_t+k})^{\gamma(1-\theta)},$$

and the lemma now follows in view of (2.2).

Our second idea is to use the refined major arc estimates for $f_j(\alpha)$ developed in Vaughan & Wooley (1991, §7). Although useful in more restricted circumstances than the estimate of Lemma 3.2, when they apply, the ensuing estimates are stronger. We define $f_j^*(\alpha)$ to be zero whenever $\alpha \notin \mathfrak{N}_j$, and by

$$f_j^*(\alpha) = Q_j(q + Q_j^k\,|\alpha q - a|)^{-1/2k} \tag{3.16}$$

whenever $\alpha \in \mathfrak{N}_j(q,a)$ and $0 \leqslant a \leqslant q \leqslant (PM_1)^{w(k-j)}$.

Lemma 3.3. *Suppose that $1 \leqslant j \leqslant k-4$. Let u be a positive integer, and define*

$$\gamma = 1 - \frac{1}{k-j+1} - \frac{2}{k+1} \quad t = [\gamma^{-1}(u-\tfrac{1}{2})+1], \quad \theta = t-\gamma^{-1}(u-\tfrac{1}{2}),$$

and
$$\sigma_u = \gamma(\theta\Delta_{t-1} + (1-\theta)\,\Delta_t) - \tfrac{1}{4}.$$

Suppose also that $u \geqslant 2\gamma k$, and $(PM_1)^{w(k-j)} \leqslant Q_j^{\frac{2}{3}}$. Then

$$I_{j,u} \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j Q_j^{2u+2-k}((PM_1)^{w(k-j)/8} Q_j^{\sigma_u} + 1).$$

Proof. Let
$$M = Q_j^{\frac{1}{2}}(PM_1)^{w(k-j)/4},$$
so that on hypothesis we have $M \geqslant (PM_1)^{w(k-j)}$. Suppose that $\alpha \in \mathfrak{N}_j$. Then by Definition 2.2, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $q \leqslant (PM_1)^{w(k-j)}$ and such that (2.5) holds. Thus by Vaughan & Wooley (1991, Lemma 7.2),
$$f_j(\alpha) \ll P^{\epsilon}(f_j^*(\alpha) + Q_j^{\frac{3}{4}}(PM_1)^{w(k-j)/8}),$$
where $f_j^*(\alpha)$ is defined by (3.16). Consequently
$$I_{j,u} \ll P^{\epsilon}(T_1 + T_2), \tag{3.17}$$
where
$$T_1 = \int_{\mathfrak{N}_j} F_j^*(\alpha) f_j^*(\alpha)\, |g_j^*(\alpha)^2 f_j(\alpha)^{2u-1}|\, d\alpha \tag{3.18}$$
and
$$T_2 = Q_j^{\frac{3}{4}}(PM_1)^{w(k-j)/8} \int_{\mathfrak{N}_j} F_j^*(\alpha)\, |g_j^*(\alpha)^2 f_j(\alpha)^{2u-1}|\, d\alpha.$$

By Hölder's inequality,
$$T_2 \ll Q_j^{\frac{3}{4}}(PM_1)^{w(k-j)/8} L_1^{1/(k-j+1)} L_2^{2/(k+1)} K_{t-1}^{\gamma\theta} K_t^{\gamma(1-\theta)},$$
where K_s $(s = t-1, t)$ and L_1, L_2 are given by (3.10), (3.11) and (3.15). Then as in the proof of the previous lemma, we obtain
$$T_2 \ll P^{1+\epsilon} \tilde{H}_j \tilde{M}_j\, Q_j^{2u+2-k+\sigma_u}(PM_1)^{w(k-j)/8}. \tag{3.19}$$
Thus if T_2 is the dominating contribution to the right-hand side of (3.17), then we are done.

Suppose then that $I_{j,u} \ll P^{\epsilon} T_1$. Applying Hölder's inequality to (3.18), we obtain
$$T_1 \ll T_3^{1/2u} I_{j,u}^{1-1/2u},$$
where
$$T_3 = \int_{\mathfrak{N}_j} F_j^*(\alpha) f_j^*(\alpha)^{2u}\, |g_j^*(\alpha)^2|\, d\alpha.$$
Then
$$I_{j,u} \ll P^{\epsilon} T_3^{1/2u} I_{j,u}^{1-1/2u},$$
and hence $I_{j,u} \ll P^{\epsilon} T_3$. A further application of Hölder's inequality now yields
$$T_3 \ll L_1^{1/(k-j+1)} L_2^{2/(k+1)} L_4^{\gamma},$$
where L_1 and L_2 are given by (3.11) and (3.15), and
$$L_4 = \int_{\mathfrak{N}_j} f_j^*(\alpha)^{2u/\gamma}\, d\alpha$$
$$= Q_j^{2u/\gamma} \sum_{q \leqslant (PM_1)^{w(k-j)}} \sum_{\substack{0 \leqslant a \leqslant q \\ (a,q)=1}} \int_{\mathfrak{N}_j(q,a)} (q + Q_j^k |\alpha q - a|)^{-u/\gamma k}\, d\alpha.$$

Then provided that $u \geqslant 2\gamma k$, we may deduce that $L_4 \ll P^{\epsilon} Q_j^{2u/\gamma - k}$, and thus, as in the proof of the previous lemma,
$$I_{j,u} \ll P^{1+\epsilon} \tilde{H}_j \tilde{M}_j\, Q_j^{2u+2-k}.$$
This completes the proof of the lemma.

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4. The iterative procedure for higher powers

We start by briefly exploring some of the consequences of the treatment of the previous section. Our aim is to show that

$$\int_0^1 |F_j(\alpha)\, g_j(\alpha)^2 f_j(\alpha)^{2u}|\, d\alpha \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j\, Q_j^{2u+2-k}((PM_1)^{-w}\, Q_j^{\Delta_{u+1}}+1), \tag{4.1}$$

which is to say that the integral in (4.1) is bounded above by the minor arc contribution together with the 'expected' major arc contribution. To provide a reasonably concise discussion, we list the following conditions. Here, and throughout §§4 and 5, ν_u, ρ_u and σ_u are defined as in the statements of Lemmata 3.1, 3.2 and 3.3.

$$(k-j+1)\, w(1+\phi_1) \leqslant (2+\Delta_{u+1}-\nu_u)\,(1-\phi_1-\ldots-\phi_j), \tag{A}$$

$$(k-j)\, w(1+\phi_1) \leqslant (2-\nu_u)\,(1-\phi_1-\ldots-\phi_j), \tag{B}$$

$$u \geqslant 2\gamma k \quad \text{and} \quad w(k-j)\,(1+\phi_1) \leqslant \tfrac{2}{3}(1-\phi_1-\ldots-\phi_j), \tag{C}$$

$$(\Delta_{u+1}-\rho_u)\,(1-\phi_1-\ldots-\phi_j) \geqslant w(1+\phi_1), \tag{α}$$

$$(\Delta_{u+1}-\sigma_u)\,(1-\phi_1-\ldots-\phi_j) \geqslant (w+\tfrac{1}{8}w(k-j))\,(1+\phi_1), \tag{β}$$

$$\rho_u = 0, \tag{γ}$$

$$\tfrac{1}{8}w(k-j)\,(1+\phi_1)+\sigma_u(1-\phi_1-\ldots-\phi_j) \leqslant 0. \tag{δ}$$

We now record a convenient consequence of Lemmata 3.1, 3.2 and 3.3.

Lemma 4.1.
(I) *Suppose that* $\Delta_{u+1}(1-\phi_1-\ldots-\phi_j) > w(1+\phi_1)$ *and condition* (A) *holds, and either condition* (α) *holds, or else both conditions* (C) *and* (β) *hold. Then*

$$\int_0^1 |F_j(\alpha)\, g_j(\alpha)^2 f_j(\alpha)^{2u}|\, d\alpha \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j\, Q_j^{2u+2-k}(PM_1)^{-w}\, Q_j^{\Delta_{u+1}}.$$

(II) *Suppose that* $\Delta_{u+1}(1-\phi_1-\ldots-\phi_j) \leqslant w(1+\phi_1)$ *and condition* (B) *holds, and either condition* (γ) *holds, or else both conditions* (C) *and* (δ) *hold. Then*

$$\int_0^1 |F_j(\alpha)\, g_j(\alpha)^2 f_j(\alpha)^{2u}|\, d\alpha \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j\, Q_j^{2u+2-k}.$$

Proof. The lemma follows in each case by simply interpreting Lemmata 3.1, 3.2 and 3.3.

The following corollary has the convenience of making no explicit reference to the ϕ_i.

Corollary. *Suppose that*

$$\Delta_{u+1}-\rho_u \geqslant w(k+1)/(k-j), \tag{4.2}$$

and

$$2+\Delta_{u+1}-\nu_u \geqslant w(k+1)\,(k-j+1)/(k-j). \tag{4.3}$$

Then
$$\int_0^1 |F_j(\alpha)\, g_j(\alpha)^2 f_j(\alpha)^{2u}|\, d\alpha \ll P^{1+\epsilon}\tilde{H}_j \tilde{M}_j\, Q_j^{2u+2-k}(PM_1)^{-w}\, Q_j^{\Delta_{u+1}}.$$

Proof. First observe that since ρ_u is non-negative, and that $0 \leqslant \phi_i \leqslant 1/k$ $(1 \leqslant i \leqslant j)$, then condition (4.2) implies that

$$\Delta_{u+1} \geqslant \Delta_{u+1} - \rho_u \geqslant \frac{w(k+1)}{k-j} \geqslant \frac{w(1+\phi_1)}{1-\phi_1-\ldots-\phi_j}.$$

Thus we are in part (I) of Lemma 4.1, and furthermore condition (α) is satisfied. Next we note that in like manner condition (4.3) implies that

$$2 + \Delta_{u+1} - \nu_u \geqslant w(k-j+1)\frac{1+\phi_1}{1-\phi_1-\ldots-\phi_j},$$

and hence condition (A) is satisfied. The corollary therefore follows from Lemma 4.1 (I).

We now turn our attention to the main task of developing the iterative procedures for higher powers based on a Hardy–Littlewood dissection. Following the notation of FIWP, our iterative procedures will be based on schemes of the following form.

$$\begin{array}{ccccccccc} F_0^2 f_0^{2s-2} & \mapsto & F_1 f_1^{2s-2} & \rightarrow & F_2 f_2^{2s-2} & \rightarrow \ldots \rightarrow & F_j f_j^{2s-2} & \Rightarrow & (F_j)\,(f_j^{2s-2}) \\ & & \downarrow & & & & \downarrow & & \\ & & f_1^{2s-2} & & & & f_{j-1}^{2s-2} & & \end{array}.$$

Suppose that the conditions of Lemma 4.1 (I) hold with $u = s-2$. Then λ_s and $\boldsymbol{\varphi}$ are determined by the equations

$$P\tilde{H}_{j-1}\tilde{M}_j Q_j^{\lambda_{s-1}} \approx P\tilde{H}_j\tilde{M}_j Q_j^{\lambda_{s-1}}(PM_1)^{-w}, \tag{4.4}$$

$$P\tilde{H}_{i-1}\tilde{M}_i Q_i^{\lambda_{s-1}} \approx (P(\tilde{H}_i\tilde{M}_i)^2 M_{i+1}^{2s-2} Q_{i+1}^{\lambda_{s-1}} Q_i^{\lambda_{s-1}})^{\frac{1}{2}} \quad (1 \leqslant i < j), \tag{4.5}$$

$$P^{\lambda_s} \approx PM_1^{2s-2} Q_1^{\lambda_{s-1}}. \tag{4.6}$$

Write $\Delta = \Delta_{s-1}$ and $\alpha = (k-\Delta)/2k$. Then following through the analysis of §13 of FIWP, we find that

$$\phi_1 = \left(\frac{1}{k+\Delta} + \left(\frac{1-w}{k} - \frac{1}{k+\Delta}\right)\alpha^{j-1}\right)\bigg/\left(1+\frac{w}{k}\alpha^{j-1}\right), \tag{4.7}$$

and that the remaining ϕ_i are given by

$$\phi_j = (1 - w(1+\phi_1))/k, \tag{4.8}$$

and
$$\phi_i = \frac{1}{k+\Delta} + \left(\phi_j - \frac{1}{k+\Delta}\right)\alpha^{j-i} \quad (1 \leqslant i < j). \tag{4.9}$$

Further, the limiting exponent λ_s^* is given by

$$\lambda_s^* = \lambda_{s-1}^*(1-\phi_1) + 1 + (2s-2)\,\phi_1. \tag{4.10}$$

Meanwhile, if the conditions of Lemma 4.1 (II) hold, then λ_s and $\boldsymbol{\varphi}$ are determined by the equations (4.5), (4.6) and

$$P\tilde{H}_{j-1}\tilde{M}_j Q_j^{\lambda_{s-1}} \approx P\tilde{H}_j\tilde{M}_j Q_j^{2s-2-k}. \tag{4.11}$$

Equations (4.5) and (4.11) lead to the equations

$$k\phi_j = 1 - \Delta(1-\phi_1-\ldots-\phi_j),$$
$$2k\phi_i = 1 + (k-\Delta)\,\phi_{i+1} \quad (1 \leqslant i < j).$$

Define a_i, b_i and c_i $(1 \leqslant i \leqslant j)$ by

$$a_j = 1-\Delta, \quad b_j = \Delta, \quad c_j = k-\Delta, \tag{4.12}$$

and

$$a_i = 1+(k-\Delta)\, a_{i+1}\, c_{i+1}^{-1}, \tag{4.13}$$

$$b_i = (k-\Delta)\, b_{i+1}\, c_{i+1}^{-1}, \tag{4.14}$$

$$c_i = 2k-(k-\Delta)\, b_{i+1}\, c_{i+1}^{-1}. \tag{4.15}$$

Then we deduce that

$$\phi_i = c_i^{-1}(a_i+b_i(\phi_1+\ldots+\phi_{i-1})) \quad (1 < i \leqslant j), \tag{4.16}$$

and

$$\phi_1 = a_1/c_1. \tag{4.17}$$

We then find that λ_s^* is again determined by (4.10).

5. The treatment for eighth powers

Henceforth we put $k = 8$. We divide into cases according to the value of s. Our analysis will be made a little simpler by noting that Δ_s may be taken to be zero when s is sufficiently large.

Lemma 5.1. *We may take* $\lambda_s = 2s-8$ *when* $s \geqslant 22$.

Proof. By reference to the appendix of FIWP, when $k=8$ we have $\lambda_{16}^* \leqslant 24.1954446$ and $\lambda_{18}^* \leqslant 28.0945483$. Let $\mathfrak{m}$ denote the set of real numbers α with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q) = 1$ and $|q\alpha - a| \leqslant P^{-7}$, then one has $q > P$. Further, define

$$f(\alpha) = \sum_{x \in \mathscr{A}(P,R)} e(\alpha x^k) \quad \text{and} \quad g(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^k).$$

Then by the argument of the proof of Vaughan (1989, Theorem 1.8), we have

$$\sup_{\alpha \in \mathfrak{m}} |f(\alpha)| \ll P^{1-\sigma+\epsilon},$$

where on recalling (2.2),

$$\sigma = (1-\Delta_{16})/64 \geqslant 0.0125711.$$

We now consider the mean value

$$\int_0^1 |g(\alpha)^2 f(\alpha)^{2s-2}|\, d\alpha$$

when $s \geqslant 22$, which, on considering the underlying diophantine equation, plainly provides an upper bound for $S_s(P,R)$. The contribution from the minor arcs $\mathfrak{m}$ is at most

$$(\sup_{\alpha \in \mathfrak{m}} |f(\alpha)|)^{2s-36} \int_0^1 |g(\alpha)^2 f(\alpha)^{34}|\, d\alpha \ll P^{(2s-36)(1-\sigma)} P^{\lambda_{18}^*+\epsilon}.$$

A little calculation reveals that $(2s-36)(1-\sigma)+\lambda_{18}^* < 2s-8$ whenever $s \geqslant 22$, and hence the minor arc contribution is acceptable. Meanwhile, the major arc contribution can be estimated satisfactorily via a standard pruning argument, owing to the presence of the classical Weyl sums $g(\alpha)$. We may therefore conclude that

$$\int_0^1 |g(\alpha)^2 f(\alpha)^{2s-2}|\, d\alpha \ll P^{2s-8},$$

and the lemma follows.

We now consider successively the values of s in the range $15 \leqslant s \leqslant 19$.

(*a*) $s = 15$

We use the Corollary to Lemma 4.1 with $j = 4$ and $u = 13$. First note that by Lemma 5.1, we have

$$\rho_{13} = \tfrac{13}{45}(\Delta_{22} + \Delta_{23}) = 0.$$

Next, by reference to the Appendix of FIWP, when $k = 8$ we may take $\lambda_{14} = 20.3659701$. Consequently

$$\Delta_{14} = 0.3659701 > 0.28125 = w(k+1)/(k-j).$$

Then (4.2) holds. Also,

$$2 + \Delta_{14} - \nu_{13} \geqslant 2 > 1.40625 = w(k+1)(k-j+1)/(k-j),$$

and so (4.3) holds. Then by the Corollary to Lemma 4.1, it follows that λ^*_{15} is given by (4.10) with ϕ_1 given by (4.7). Thus we obtain $\phi_1 \leqslant 0.11822800$ and $\lambda^*_{15} \leqslant 22.2685262$.

(*b*) $s = 16$

We use Lemma 4.1 with $j = 4$ and $u = 14$. As in the case $s = 15$, we find that $\rho_{14} = 0$, and that condition (A) (which follows from (4.3)) is satisfied easily. Then provided only that (α) holds, that is

$$\Delta_{15}(1 - \phi_1 - \ldots - \phi_4) \geqslant \tfrac{1}{8}(1 + \phi_1),$$

we may deduce that the ϕ_i are give by (4.7), (4.8) and (4.9). A calculation reveals that the latter equations give $\phi_1 \leqslant 0.119425 05$, $\phi_2 \leqslant 0.11780429$, $\phi_3 \leqslant 0.11445019$, and $\phi_4 \leqslant 0.10750899$. Thus the desired condition is indeed met, and by (4.10) we have $\lambda^*_{16} \leqslant 24.1918579$.

(*c*) $s = 17$

We use the Corollary to Lemma 4.1 with $j = 3$ and $u = 15$. We find once again that $\rho_{15} = 0$, and that (4.3) is satisfied easily. Condition (4.2) is also satisfied, since by using the conclusion of part (*b*), we have

$$\Delta_{16} = 0.1918579 > 0.1125 = w(k+1)/(k-j).$$

Hence we may deduce that the ϕ_i are given by (4.7), (4.8) and (4.9). A calculation reveals that $\phi_1 \leqslant 0.12068453$, and hence by (4.10), $\lambda^*_{17} \leqslant 26.1341799$.

(*d*) $s = 18$

We use the Corollary to Lemma 4.1 with $j = 3$ and $u = 16$. Once more, we find that $\rho_{16} = 0$, and we have

$$\Delta_{17} = 0.1341799 > 0.1125 = w(k+1)/(k-j).$$

Thus conditions (4.2) and (4.3) are satisfied easily. The ϕ_i are therefore given by (4.7), (4.8) and (4.9). Then $\phi_1 \leqslant 0.12131915$, and hence by (4.10), $\lambda^*_{18} \leqslant 28.0884545$.

(*e*) $s = 19$

We use the Corollary to Lemma 4.1 with $j = 2$ and $u = 17$. Again we find that $\rho_{17} = 0$. Also

$$\Delta_{18} = 0.0884545 > 0.046875 \geqslant w(k+1)/(k-j).$$

Hence conditions (4.2) and (4.3) are met, and we may deduce that ϕ_i are given by (4.7), (4.8) and (4.9). Then we have $\phi_1 \leqslant 0.12214150$, and hence by (4.10), $\lambda^*_{19} \leqslant 30.0547826$.

We now complete the proof of the theorem. Let $X = P^{8/15}$ and $Z = PX^{-1}$. Define the generating function $h(\alpha)$ by

$$h(\alpha) = \sum_{x \in \mathscr{C}} e(\alpha x^8),$$

where $$\mathscr{C} = \{x : x = pz,\ X/2 < p \leqslant X,\ p \text{ prime},\ z \in \mathscr{A}(Z, Z^\eta)\}.$$

Let s be an even integer, and write $s = 2r$. Define $\mathfrak{m}$ to be the set of real numbers α in $(\frac{1}{16}P^{-7}, 1+\frac{1}{16}P^{-7}]$ with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q) = 1$ and $|\alpha - a/q| \leqslant q^{-1}X^{-7}(rZ^8)^{-1}$, then one has $q > X$. Then the argument of Vaughan (1989, §9) gives

$$\sup_{\alpha \in \mathfrak{m}} |h(\alpha)| \ll P^{1-\sigma+\epsilon}, \tag{5.1}$$

where $$\sigma = (8 - 7\Delta_s)/30s. \tag{5.2}$$

By (5.2) with $s = 16$, we obtain $\sigma > 0.01386873$. Moreover $\lambda^*_{19} + 4(1-\sigma) < 34$. Then by Vaughan & Wooley (1991, Theorem 4), we may finally conclude that $G(8) \leqslant 42$.

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